MATH 031 – Spring 2026 Worksheet #15

Elementary Row Matrices: Detailed Solutions

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This document is an independent educational solution guide. It is not an official answer key from the instructor, department, or university. Students should use this document as a learning resource and should verify all answers with their instructor, textbook, and class notes. The purpose is to explain the reasoning clearly and help students understand elementary row matrices and row-reduction factorization.

Main Idea: Elementary Row Matrices

When we perform row operations on a matrix A, each row operation can be represented by left multiplication by an elementary matrix.

$$A \longrightarrow E_1A \longrightarrow E_2E_1A \longrightarrow E_3E_2E_1A \longrightarrow \cdots$$

Therefore, if we row-reduce A to its reduced row echelon form, then we can write

$$E_kE_{k-1}\cdots E_2E_1A = \mathrm{RREF}(A).$$

The order is important. The first row operation corresponds to E_1, but in the final product it appears closest to A.

Exercise 1

Obtain a factorization

$$E_k \cdots E_2E_1A = \mathrm{RREF}(A)$$

for each matrix.

Exercise 1(a)

We are given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We row-reduce A to its reduced row echelon form.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 := R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

The elementary matrix for the operation $R_2 := R_2 - 3R_1$ is obtained by applying the same row operation to I_2:

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Thus,

$$E_1 A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

Next, make the second pivot equal to 1:

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2 := -\frac{1}{2} R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrix is

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Then

$$E_2 E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Finally, eliminate the entry above the pivot in column 2:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 := R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrix is

$$E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{RREF}(A).$$

So one valid factorization is

$$\boxed{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Hence,

$$\boxed{\mathrm{RREF}(A) = I_2.}$$

Exercise 1(b)

We are given

$$B = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix}.$$

We row-reduce B.
First, eliminate the entry below the first pivot in row 2:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 := R_2 - R_1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}.$$

The elementary matrix is

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, eliminate the entry below the first pivot in row 3:

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 := R_3 - R_1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

The elementary matrix is

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Now eliminate the entry below the second pivot:

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 := R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary matrix is

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Finally, eliminate the entry above the second pivot:

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 := R_1 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary matrix is

$$E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$E_4 E_3 E_2 E_1 B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(B).$$

So one valid factorization is

$$\boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Hence,

$$\boxed{\text{RREF}(B) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.}$$

Exercise 1(c)

We are given

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix}.$$

We row-reduce C.
First, eliminate the entry below the first pivot in row 2:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow{R_2 := R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix}.$$

The elementary matrix is

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, eliminate the entry below the first pivot in row 3:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow{R_3 := R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{bmatrix}.$$

The elementary matrix is

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Now eliminate the entry below the second pivot:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 := R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary matrix is

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}.$$

The matrix is now in reduced row echelon form because the pivot columns already have zeros above and below the pivots.

Therefore,

$$E_3 E_2 E_1 C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{RREF}(C).$$

So one valid factorization is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Hence,

$$\boxed{\mathrm{RREF}(C) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.}$$

Exercise 2

Find the determinants of all elementary row matrices of sizes 2×2 and 3×3.

General Pattern

There are three types of elementary row matrices.

Type I. Switching two rows.

This changes the sign of the determinant. Therefore,

$$\det(E) = -1.$$

Type II. Multiplying one row by a nonzero scalar α.

This multiplies the determinant by α. Therefore,

$$\det(E) = \alpha.$$

Type III. Adding a multiple of one row to another row.

This does not change the determinant. Therefore,

$$\det(E) = 1.$$

All Elementary Row Matrices of Size 2×2

Type I: Switching Rows

The only possible row-switching matrix of size 2×2 is

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Its determinant is

$$\det(E) = (0)(0) - (1)(1) = -1.$$

Therefore,

$$\boxed{\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.}$$

Type II: Multiplying One Row by α

There are two possible matrices:

$$E_1 = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0.$$

Their determinants are

$$\det(E_1) = \alpha \cdot 1 - 0 \cdot 0 = \alpha,$$

and

$$\det(E_2) = 1 \cdot \alpha - 0 \cdot 0 = \alpha.$$

Therefore,

$$\boxed{\det \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} = \alpha}$$

and

$$\boxed{\det \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} = \alpha.}$$

Type III: Adding a Multiple of One Row to Another

There are two possible matrices:

$$E_1 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

Their determinants are

$$\det(E_1) = (1)(1) - (0)(c) = 1,$$

and

$$\det(E_2) = (1)(1) - (c)(0) = 1.$$

Therefore,

$$\boxed{\det \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = 1}$$

and

$$\boxed{\det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1.}$$

All Elementary Row Matrices of Size 3×3

Type I: Switching Two Rows

There are three possible row switches.

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Each matrix is obtained from I_3 by switching two rows. A single row switch changes the sign of the determinant. Since

$$\det(I_3) = 1,$$

each type I elementary matrix has determinant

$$-1.$$

Therefore,

$$\boxed{\det(E_{12}) = \det(E_{13}) = \det(E_{23}) = -1.}$$

Explicitly,

$$\boxed{\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1}$$

$$\boxed{\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1}$$

and

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = -1.}$$

Type II: Multiplying One Row by α

There are three possible matrices:

$$E_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0.$$

Since each matrix is diagonal, the determinant is the product of the diagonal entries. Thus,

$$\det(E_1) = \alpha \cdot 1 \cdot 1 = \alpha,$$

$$\det(E_2) = 1 \cdot \alpha \cdot 1 = \alpha,$$

and

$$\det(E_3) = 1 \cdot 1 \cdot \alpha = \alpha.$$

Therefore,

$$\boxed{\det \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \alpha}$$

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \alpha}$$

and

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \alpha.}$$

Type III: Adding a Multiple of One Row to Another

For a 3×3 matrix, there are six possible ordered row replacement operations because we can choose $R_i := R_i + cR_j$ with $i \neq j$.

They are:

$$R_1 := R_1 + cR_2 : \qquad E_{12} = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_1 := R_1 + cR_3 : \qquad E_{13} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_2 := R_2 + cR_1 : \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_2 := R_2 + cR_3 : \qquad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3 := R_3 + cR_1 : \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix},$$

and

$$R_3 := R_3 + cR_2 : \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Each of these matrices is triangular with diagonal entries all equal to 1. Therefore, each determinant is the product of the diagonal entries:

$$1 \cdot 1 \cdot 1 = 1.$$

Therefore,

$$\boxed{\det(E_{12}) = \det(E_{13}) = \det(E_{21}) = \det(E_{23}) = \det(E_{31}) = \det(E_{32}) = 1.}$$

Explicitly,

$$\boxed{\det \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1}$$

$$\boxed{\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1}$$

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1}$$

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = 1}$$

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} = 1}$$

and

$$\boxed{\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = 1.}$$

Summary of Exercise 2

For any elementary row matrix E:

Type	Row Operation	$\det(E)$
Type I	Switch two rows	-1
Type II	Multiply one row by $\alpha \neq 0$	α
Type III	Add a multiple of one row to another row	1

Final Answers

Exercise 1

$$\boxed{\text{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\boxed{\text{RREF}(B) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boxed{\text{RREF}(C) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Exercise 2

$$\boxed{\det(\text{Type I elementary matrix}) = -1}$$

$$\boxed{\det(\text{Type II elementary matrix}) = \alpha}$$

$$\boxed{\det(\text{Type III elementary matrix}) = 1}$$