Detailed Solutions to Worksheet #17

Eigenvalues and Eigenvectors

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Throughout this solution, we use the characteristic polynomial convention

$$p_A(\lambda) = \det(\lambda I - A).$$

Exercise 1

Find the characteristic polynomial and the eigenvalues of the following matrices:

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}.$$

Which of them are diagonalizable?

Matrix A

We compute

$$p_A(\lambda) = \det(\lambda I - A).$$

Thus,

$$\lambda I - A = \begin{bmatrix} \lambda - 2 & -7 \\ -7 & \lambda - 2 \end{bmatrix}.$$

Therefore,

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 2 & -7 \\ -7 & \lambda - 2 \end{bmatrix}.$$

Using the determinant formula for a 2×2 matrix,

$$p_A(\lambda) = (\lambda - 2)(\lambda - 2) - (-7)(-7).$$

Hence,

$$p_A(\lambda) = (\lambda - 2)^2 - 49.$$

So

$$p_A(\lambda) = \lambda^2 - 4\lambda + 4 - 49 = \lambda^2 - 4\lambda - 45.$$

Factoring,

$$p_A(\lambda) = (\lambda - 9)(\lambda + 5).$$

Therefore, the eigenvalues are

$$\boxed{\lambda = 9} \qquad \text{and} \qquad \boxed{\lambda = -5}.$$

Since A has two distinct real eigenvalues, it is diagonalizable over $\mathbb{R}$.

$$\boxed{A \text{ is diagonalizable.}}$$

Matrix B

We compute

$$p_B(\lambda) = \det(\lambda I - B).$$

We have

$$\lambda I - B = \begin{bmatrix} \lambda - 3 & 2 \\ -1 & \lambda + 1 \end{bmatrix}.$$

Therefore,

$$p_B(\lambda) = \det \begin{bmatrix} \lambda - 3 & 2 \\ -1 & \lambda + 1 \end{bmatrix}.$$

Thus,

$$p_B(\lambda) = (\lambda - 3)(\lambda + 1) - 2(-1).$$

Simplifying,

$$p_B(\lambda) = (\lambda - 3)(\lambda + 1) + 2.$$

Now,

$$(\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3.$$

Therefore,

$$p_B(\lambda) = \lambda^2 - 2\lambda - 3 + 2 = \lambda^2 - 2\lambda - 1.$$

Now solve

$$\lambda^2 - 2\lambda - 1 = 0.$$

Using the quadratic formula,

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{4 + 4}}{2}.$$

So

$$\lambda = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2}.$$

Therefore,

$$\lambda = 1 \pm \sqrt{2}.$$

The eigenvalues are

$$\boxed{\lambda = 1 + \sqrt{2}} \qquad \text{and} \qquad \boxed{\lambda = 1 - \sqrt{2}}.$$

Since B has two distinct real eigenvalues, it is diagonalizable over $\mathbb{R}$.

$$\boxed{B \text{ is diagonalizable.}}$$

Matrix C

We compute

$$p_C(\lambda) = \det(\lambda I - C).$$

We have

$$\lambda I - C = \begin{bmatrix} \lambda - 1 & 4 \\ -4 & \lambda - 6 \end{bmatrix}.$$

Therefore,

$$p_C(\lambda) = \det \begin{bmatrix} \lambda - 1 & 4 \\ -4 & \lambda - 6 \end{bmatrix}.$$

So

$$p_C(\lambda) = (\lambda - 1)(\lambda - 6) - 4(-4).$$

Thus,

$$p_C(\lambda) = (\lambda - 1)(\lambda - 6) + 16.$$

Now,

$$(\lambda - 1)(\lambda - 6) = \lambda^2 - 7\lambda + 6.$$

Therefore,

$$p_C(\lambda) = \lambda^2 - 7\lambda + 6 + 16 = \lambda^2 - 7\lambda + 22.$$

Now solve

$$\lambda^2 - 7\lambda + 22 = 0.$$

The discriminant is

$$\Delta = (-7)^2 - 4(1)(22) = 49 - 88 = -39.$$

Since the discriminant is negative, the eigenvalues are complex Over $\mathbb{R}$, this matrix is not diagonalizable because it has no real eigenvalues.

$$\boxed{C \text{ is not diagonalizable over } \mathbb{R},}$$

Matrix D

We compute

$$p_D(\lambda) = \det(\lambda I - D).$$

We have

$$\lambda I - D = \begin{bmatrix} \lambda - 7 & 2 \\ -2 & \lambda - 3 \end{bmatrix}.$$

Therefore,

$$p_D(\lambda) = \det \begin{bmatrix} \lambda - 7 & 2 \\ -2 & \lambda - 3 \end{bmatrix}.$$

Thus,

$$p_D(\lambda) = (\lambda - 7)(\lambda - 3) - 2(-2).$$

So

$$p_D(\lambda) = (\lambda - 7)(\lambda - 3) + 4.$$

Now,

$$(\lambda - 7)(\lambda - 3) = \lambda^2 - 10\lambda + 21.$$

Therefore,

$$p_D(\lambda) = \lambda^2 - 10\lambda + 21 + 4 = \lambda^2 - 10\lambda + 25.$$

Factoring,

$$p_D(\lambda) = (\lambda - 5)^2.$$

Thus the only eigenvalue is

$$\boxed{\lambda = 5}$$

with algebraic multiplicity 2.

Now we check the eigenspace. We solve

$$(D - 5I)x = 0.$$

We have

$$D - 5I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}.$$

So the system is

$$2x - 2y = 0,$$

which gives

$$x = y.$$

Therefore, the eigenspace is

$$E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace has dimension 1 while dimension of matrix is 2. Therefore, D does not have enough linearly independent eigenvectors to be diagonalizable.

$$\boxed{D \text{ is not diagonalizable.}}$$

Summary for Exercise 1

Matrix	Characteristic Polynomial	Eigenvalues	Diagonalizable over $\mathbb{R}$?
A	$(\lambda-9)(\lambda+5)$	$9, -5$	Yes
B	$\lambda^2-2\lambda-1$	$1+\sqrt{2}, 1-\sqrt{2}$	Yes
C	$\lambda^2-7\lambda+22$	not real	No
D	$(\lambda-5)^2$	5	No

Exercise 2

Now do the same for the following 3×3 matrices:

$$A=\begin{bmatrix}1&0&-1\\2&3&-1\\0&6&0\end{bmatrix},\qquad B=\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix}.$$

Diagonalize them if possible.

Matrix A

We compute

$$p_A(\lambda)=\det(\lambda I-A).$$

We have

$$\lambda I-A=\begin{bmatrix}\lambda-1&0&1\\-2&\lambda-3&1\\0&-6&\lambda\end{bmatrix}.$$

Therefore,

$$p_A(\lambda)=\det\begin{bmatrix}\lambda-1&0&1\\-2&\lambda-3&1\\0&-6&\lambda\end{bmatrix}.$$

Expanding along the first row,

$$p_A(\lambda)=(\lambda-1)\det\begin{bmatrix}\lambda-3&1\\-6&\lambda\end{bmatrix}+1\det\begin{bmatrix}-2&\lambda-3\\0&-6\end{bmatrix}.$$

Now,

$$\det\begin{bmatrix}\lambda-3&1\\-6&\lambda\end{bmatrix}=(\lambda-3)\lambda-1(-6)=\lambda^2-3\lambda+6.$$

Also,

$$\det\begin{bmatrix}-2&\lambda-3\\0&-6\end{bmatrix}=(-2)(-6)-0(\lambda-3)=12.$$

Thus,

$$p_A(\lambda)=(\lambda-1)(\lambda^2-3\lambda+6)+12.$$

Expanding,

$$(\lambda - 1)(\lambda^2 - 3\lambda + 6) = \lambda^3 - 4\lambda^2 + 9\lambda - 6.$$

Therefore,

$$p_A(\lambda) = \lambda^3 - 4\lambda^2 + 9\lambda - 6 + 12.$$

So

$$\boxed{p_A(\lambda) = \lambda^3 - 4\lambda^2 + 9\lambda + 6.}$$

This cubic does not have rational roots. Numerically, the eigenvalues are approximately

$$\boxed{\lambda_1 \approx -0.526980}$$

and

$$\boxed{\lambda_{2,3} \approx 2.263490 \pm 2.502447i.}$$

Since only one eigenvalue is real and the other two are complex, this matrix is not diagonalizable over $\mathbb{R}$.

$$\boxed{A \text{ is not diagonalizable over } \mathbb{R}.}$$

Matrix B

Now consider

$$B = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

We compute

$$p_B(\lambda) = \det(\lambda I - B).$$

We have

$$\lambda I - B = \begin{bmatrix} \lambda & -3 & -1 \\ -3 & \lambda & -2 \\ -1 & -2 & \lambda \end{bmatrix}.$$

Therefore,

$$p_B(\lambda) = \det \begin{bmatrix} \lambda & -3 & -1 \\ -3 & \lambda & -2 \\ -1 & -2 & \lambda \end{bmatrix}.$$

Expanding along the first row,

$$p_B(\lambda) = \lambda \det \begin{bmatrix} \lambda & -2 \\ -2 & \lambda \end{bmatrix} - (-3) \det \begin{bmatrix} -3 & -2 \\ -1 & \lambda \end{bmatrix} + (-1) \det \begin{bmatrix} -3 & \lambda \\ -1 & -2 \end{bmatrix}.$$

Now,

$$\det \begin{bmatrix} \lambda & -2 \\ -2 & \lambda \end{bmatrix} = \lambda^2 - 4.$$

Also,

$$\det \begin{bmatrix} -3 & -2 \\ -1 & \lambda \end{bmatrix} = (-3)\lambda - (-2)(-1) = -3\lambda - 2.$$

And

$$\det \begin{bmatrix} -3 & \lambda \\ -1 & -2 \end{bmatrix} = (-3)(-2) - \lambda(-1) = 6 + \lambda.$$

Therefore,

$$p_B(\lambda) = \lambda(\lambda^2 - 4) + 3(-3\lambda - 2) - (6 + \lambda).$$

Simplifying,

$$p_B(\lambda) = \lambda^3 - 4\lambda - 9\lambda - 6 - 6 - \lambda.$$

Thus,

$$p_B(\lambda) = \lambda^3 - 14\lambda - 12.$$

So

$$\boxed{p_B(\lambda) = \lambda^3 - 14\lambda - 12.}$$

This polynomial has three distinct real roots. Numerically, the eigenvalues are

$$\boxed{\lambda_1 \approx 4.113091}, \qquad \boxed{\lambda_2 \approx -0.911179}, \qquad \boxed{\lambda_3 \approx -3.201912}.$$

Because B has three distinct real eigenvalues, it is diagonalizable over $\mathbb{R}$.

$$\boxed{B \text{ is diagonalizable over } \mathbb{R}.}$$

In fact, B is symmetric since

$$B^T = B.$$

Therefore, by the Spectral Theorem, B is orthogonally diagonalizable over $\mathbb{R}$.

For a general eigenvalue λ, solve

$$(B - \lambda I)v = 0.$$

A convenient eigenvector corresponding to λ is

$$v(\lambda) = \begin{bmatrix} \lambda + 6 \\ 2\lambda + 3 \\ \lambda^2 - 9 \end{bmatrix}.$$

Indeed,

$$Bv(\lambda) = \lambda v(\lambda)$$

whenever

$$\lambda^3 - 14\lambda - 12 = 0.$$

Therefore, if $\lambda_1, \lambda_2, \lambda_3$ are the three real roots of

$$\lambda^3 - 14\lambda - 12 = 0,$$

then

$$B = PDP^{-1},$$

where

$$P = \begin{bmatrix} | & | & | \\ v(\lambda_1) & v(\lambda_2) & v(\lambda_3) \\ | & | & | \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Explicitly,

$$P = \begin{bmatrix} \lambda_1 + 6 & \lambda_2 + 6 & \lambda_3 + 6 \\ 2\lambda_1 + 3 & 2\lambda_2 + 3 & 2\lambda_3 + 3 \\ \lambda_1^2 - 9 & \lambda_2^2 - 9 & \lambda_3^2 - 9 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are linearly independent, so P is invertible.

Exercise 3

Find A^{2026}, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

First compute A^2:

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Multiplying,

$$A^2 = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Thus,

$$A^2 = 2A.$$

Now we use this pattern. Since

$$A^2 = 2A,$$

then

$$A^3 = A^2 A = (2A)A = 2A^2 = 2(2A) = 2^2 A.$$

Similarly,

$$A^4 = A^3 A = (2^2 A)A = 2^2 A^2 = 2^2(2A) = 2^3 A.$$

Therefore, by induction,

$$A^n = 2^{n-1} A$$

for every integer $n \geq 1$.

Thus,

$$A^{2026} = 2^{2025} A.$$

Therefore,

$$\boxed{A^{2026} = 2^{2025} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$$

or equivalently,

$$\boxed{A^{2026} = \begin{bmatrix} 2^{2025} & 2^{2025} \\ 2^{2025} & 2^{2025} \end{bmatrix}.}$$

Exercise 4

True or false? Briefly justify your answers.

(i)

Statement:

$$\text{If } Ax = \lambda x \text{ for some vector } x, \text{ then } \lambda \text{ is an eigenvalue of } A.$$

This statement is **false** as written.
The reason is that an eigenvector must be nonzero. If $x = 0$, then

$$A0 = \lambda 0$$

is true for every scalar λ. But that does not mean every scalar λ is an eigenvalue.
The correct statement should be:

$$\text{If } Ax = \lambda x \text{ for some nonzero vector } x, \text{ then } \lambda \text{ is an eigenvalue of } A.$$

Thus,

$$\boxed{\text{False.}}$$

(ii)

Statement:

A matrix is invertible if and only if it has zero as an eigenvalue.

This statement is **false**.
A square matrix A is invertible if and only if

$$\det(A) \neq 0.$$

Also, 0 is an eigenvalue of A if and only if

$$\det(A - 0I) = \det(A) = 0.$$

Therefore, A has 0 as an eigenvalue exactly when A is not invertible.
So the correct statement is:

A is invertible if and only if 0 is not an eigenvalue of A.

Thus,

$$\boxed{\text{False.}}$$

(iii)

Statement:

If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

This statement is **false**.

Linearly independent eigenvectors can correspond to the same eigenvalue. For example, consider the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Every nonzero vector is an eigenvector of I with eigenvalue 1. For instance,

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly independent, but both correspond to the same eigenvalue $\lambda = 1$.

Thus,

$$\boxed{\text{False.}}$$

(iv)

Statement:

If v is an eigenvector with eigenvalue 2, then $2v$ is an eigenvector with eigenvalue 4.

This statement is **false**.

If v is an eigenvector with eigenvalue 2, then

$$Av = 2v.$$

Now consider $2v$. We compute:

$$A(2v) = 2Av.$$

Since $Av = 2v$, we get

$$A(2v) = 2(2v) = 4v.$$

But

$$4v = 2(2v).$$

Therefore,

$$A(2v) = 2(2v).$$

So $2v$ is still an eigenvector with eigenvalue 2, not eigenvalue 4.

Thus,

$$\boxed{\text{False.}}$$

(v)

Statement:

$$\text{The characteristic polynomials of } A \text{ and } A^T \text{ are equal, and so they have the same eigenvalues}$$
$$\text{but not necessarily the same eigenvectors.}$$

This statement is **true**.

We have

$$p_{A^T}(\lambda) = \det(\lambda I - A^T).$$

Notice that

$$\lambda I - A^T = (\lambda I - A)^T.$$

Therefore,

$$p_{A^T}(\lambda) = \det\left((\lambda I - A)^T\right).$$

Since the determinant of a matrix equals the determinant of its transpose,

$$\det(M^T) = \det(M),$$

we get

$$p_{A^T}(\lambda) = \det(\lambda I - A) = p_A(\lambda).$$

Thus, A and A^T have the same characteristic polynomial and therefore the same eigenvalues.

However, they do not necessarily have the same eigenvectors.

Thus,

$$\boxed{\text{True.}}$$

Summary for Exercise 4

Statement	Answer
(i)	False
(ii)	False
(iii)	False
(iv)	False
(v)	True