Detailed Solutions to Worksheet #16

MATH 031 Applied Linear Algebra

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Disclaimer

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Exercise 1

Use cofactor expansions to evaluate the determinants.

Exercise 1(a)

Evaluate

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 6 \end{vmatrix}.$$

Let

$$A = \begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 6 \end{bmatrix}.$$

We want to use cofactor expansion. The best row to choose is row 3 because it contains three zeros:

$$\text{Row } 3 = \begin{bmatrix} 3 & 0 & 0 & 0 \end{bmatrix}.$$

Expanding along row 3, we get

$$\det(A) = 3(-1)^{3+1} \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix}.$$

Since

$$(-1)^{3+1} = (-1)^4 = 1,$$

we have

$$\det(A) = 3 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix}.$$

Now evaluate the 3×3 determinant:

$$\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix}.$$

Again, expand along the first row because it has two zeros:

$$\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix} = 5(-1)^{1+3} \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}.$$

Since

$$(-1)^{1+3} = (-1)^4 = 1,$$

we get

$$\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix} = 5 \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}.$$

Compute the 2×2 determinant:

$$\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 7(1) - 2(3) = 7 - 6 = 1.$$

Therefore,

$$\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 6 \end{vmatrix} = 5(1) = 5.$$

Thus,

$$\det(A) = 3(5) = 15.$$

So the answer is

$$\boxed{15}.$$

Exercise 1(b)

Evaluate

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}.$$

Let

$$B = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}.$$

We use cofactor expansion. The best row to choose is row 2 because it has only one nonzero entry:

$$\text{Row } 2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

The nonzero entry is 2 in position $(2, 3)$. Therefore,

$$\det(B) = 2(-1)^{2+3} \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}.$$

Since

$$(-1)^{2+3} = (-1)^5 = -1,$$

we get

$$\det(B) = -2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}.$$

Now let

$$C = \begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

We evaluate $\det(C)$. Notice that column 2 has only one nonzero entry:

$$\text{Column } 2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Expanding along column 2:

$$\det(C) = 3(-1)^{2+2} \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}.$$

Since

$$(-1)^{2+2} = (-1)^4 = 1,$$

we have

$$\det(C) = 3 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}.$$

Now compute the 3×3 determinant:

$$\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}.$$

Using cofactor expansion along the first row:

$$\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = 4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 5 & -3 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}.$$

Compute each 2×2 determinant:

$$\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = 2(2) - (-3)(-1) = 4 - 3 = 1,$$

$$\begin{vmatrix} 5 & -3 \\ 0 & 2 \end{vmatrix} = 5(2) - (-3)(0) = 10,$$

and

$$\begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} = 5(-1) - 2(0) = -5.$$

Therefore,

$$\begin{aligned} \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} &= 4(1) - 3(10) + (-5)(-5) \\ &= 4 - 30 + 25 \\ &= -1. \end{aligned}$$

So

$$\det(C) = 3(-1) = -3.$$

Finally,

$$\det(B) = -2\det(C) = -2(-3) = 6.$$

Thus, the answer is

$$\boxed{6}.$$

Exercise 2

Compute the determinants by row reduction to echelon form.

Important Determinant Facts for Row Reduction

We will use the following facts:

1. A row replacement operation, such as

$$R_i \leftarrow R_i + cR_j,$$

does not change the determinant.

2. If a matrix is triangular, then its determinant is the product of its diagonal entries.
3. If a row becomes zero during row reduction, then the determinant is zero.

Exercise 2(a)

Compute

$$\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & 8 & 7 \end{vmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & 8 & 7 \end{bmatrix}.$$

We use row replacement operations to convert A into echelon form.
First, eliminate the entries below the first pivot:

$$R_2 \leftarrow R_2 + R_1,$$

$$R_3 \leftarrow R_3 + 2R_1.$$

This gives

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 18 & -1 \end{bmatrix}.$$

Now eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 - 18R_2.$$

Then

$$R_3 = \begin{bmatrix} 0 & 18 & -1 \end{bmatrix} - 18 \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -19 \end{bmatrix}.$$

So the echelon form is

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -19 \end{bmatrix}.$$

Because we used only row replacement operations, the determinant did not change. The resulting matrix is upper triangular, so its determinant is the product of the diagonal entries:

$$\det(A) = 1(1)(-19) = -19.$$

Therefore,

$$\boxed{-19}.$$

Exercise 2(b)

Compute

$$\begin{vmatrix} 2 & 2 & -2 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix}.$$

Let

$$B = \begin{bmatrix} 2 & 2 & -2 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix}.$$

We row reduce to echelon form using row replacements.
First, eliminate the entries below the first pivot.
Use

$$R_2 \leftarrow R_2 - \frac{3}{2}R_1.$$

Then

$$R_2 = \begin{bmatrix} 3 & 4 & -4 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}.$$

Next use

$$R_3 \leftarrow R_3 - R_1.$$

Then

$$R_3 = \begin{bmatrix} 2 & -3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \end{bmatrix}.$$

So we have

$$\begin{bmatrix} 2 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{bmatrix}.$$

Now eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 + 5R_2.$$

Then

$$R_3 = \begin{bmatrix} 0 & -5 & -3 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -8 \end{bmatrix}.$$

Thus the echelon form is

$$\begin{bmatrix} 2 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}.$$

Only row replacement operations were used, so the determinant is unchanged. Since the matrix is upper triangular,

$$\det(B) = 2(1)(-8) = -16.$$

Therefore,

$$\boxed{-16}.$$

Exercise 2(c)

Compute

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -2 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}.$$

Let

$$C = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -2 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}.$$

We row reduce using row replacement operations.
First, eliminate the entries below the first pivot:

$$R_2 \leftarrow R_2 + 2R_1,$$

$$R_3 \leftarrow R_3 - 3R_1,$$

$$R_4 \leftarrow R_4 - R_1.$$

Compute each row:

$$R_2 = \begin{bmatrix} -2 & -2 & 7 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 7 & 8 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 3 & 5 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 2 & -5 \end{bmatrix},$$

and

$$R_4 = \begin{bmatrix} 1 & -1 & 2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 2 & -5 \end{bmatrix}.$$

So the matrix becomes

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 4 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{bmatrix}.$$

Notice that rows 3 and 4 are now identical:

$$R_3 = R_4.$$

If two rows of a matrix are equal, then the determinant is zero. Therefore,

$$\det(C) = 0.$$

To see this by continuing row reduction, we can also do:

$$R_4 \leftarrow R_4 - R_3.$$

This gives a zero row:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 4 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the echelon form has a zero row, the determinant must be zero.
Thus,

$$\boxed{0}.$$

Exercise 3

Let A and B be 3×3 matrices, with

$$\det(A) = -2$$

and

$$\det(B) = 3.$$

Use the standard properties of determinants to find

$$\det(AB), \qquad \det(5A), \qquad \det(B^T), \qquad \det(A^{-1}), \qquad \det(A^3).$$

1. Finding $\det(AB)$

We use the determinant property

$$\det(AB) = \det(A)\det(B).$$

Therefore,

$$\det(AB) = (-2)(3) = -6.$$

So

$$\boxed{\det(AB) = -6}.$$

2. Finding $\det(5A)$

Since A is a 3×3 matrix, multiplying the matrix by a scalar 5 multiplies the determinant by 5^3.

In general, for an $n \times n$ matrix,

$$\det(cA) = c^n \det(A).$$

Here, $n = 3$ and $c = 5$, so

$$\det(5A) = 5^3 \det(A).$$

Thus,

$$\det(5A) = 125(-2) = -250.$$

So

$$\boxed{\det(5A) = -250}.$$

3. Finding $\det(B^T)$

We use the determinant property

$$\det(B^T) = \det(B).$$

Therefore,

$$\det(B^T) = 3.$$

So

$$\boxed{\det(B^T) = 3}.$$

4. Finding $\det(A^{-1})$

We use the determinant property

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Since

$$\det(A) = -2,$$

we have

$$\det(A^{-1}) = \frac{1}{-2} = -\frac{1}{2}.$$

So

$$\boxed{\det(A^{-1}) = -\frac{1}{2}}.$$

5. Finding $\det(A^3)$

We use the property

$$\det(A^3) = \det(A \cdot A \cdot A).$$

Using the multiplication property of determinants,

$$\det(A^3) = \det(A)\det(A)\det(A) = (\det(A))^3.$$

Since

$$\det(A) = -2,$$

we get

$$\det(A^3) = (-2)^3 = -8.$$

So

$$\boxed{\det(A^3) = -8}.$$

Final Answers for Exercise 3

$$\boxed{\det(AB) = -6}$$

$$\boxed{\det(5A) = -250}$$

$$\boxed{\det(B^T) = 3}$$

$$\boxed{\det(A^{-1}) = -\frac{1}{2}}$$

$$\boxed{\det(A^3) = -8}$$

Exercise 4

True or false? Briefly justify your answers.

(i) A row replacement operation does not affect the determinant of a matrix.

This statement is true.

A row replacement operation has the form

$$R_i \leftarrow R_i + cR_j,$$

where $i \neq j$.

This operation adds a multiple of one row to another row. It does not change the determinant.

Therefore,

$$\boxed{\text{True}}.$$

(ii) If $\det(A)$ is zero, then two rows or two columns are the same, or a row or a column is zero.

This statement is false.

It is true that if a matrix has two equal rows, two equal columns, a zero row, or a zero column, then its determinant is zero. However, the converse is not always true.

A determinant can be zero simply because the rows or columns are linearly dependent, even if no two rows are equal and no row is zero.

For example, consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Then

$$\det(A) = 1(4) - 2(2) = 4 - 4 = 0.$$

However, the two rows are not the same:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \end{bmatrix},$$

and neither row is zero.

The reason the determinant is zero is that the second row is a multiple of the first row:

$$\begin{bmatrix} 2 & 4 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Therefore,

$$\boxed{\text{False}}.$$

(iii) The determinant of A is the product of the diagonal entries of A.

This statement is false in general.

The determinant is the product of the diagonal entries only for triangular matrices, such as upper triangular or lower triangular matrices.

For example, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The product of the diagonal entries is

$$1 \cdot 4 = 4.$$

But the determinant is

$$\det(A) = 1(4) - 2(3) = 4 - 6 = -2.$$

Since

$$\det(A) \neq 4,$$

the determinant is not always the product of the diagonal entries.

Therefore,

$$\boxed{\text{False}}.$$

(iv) $\det(A + B) = \det(A) + \det(B)$.

This statement is false.

The determinant is not additive. In general,

$$\det(A + B) \neq \det(A) + \det(B).$$

For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Then

$$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2.$$

Thus,

$$\det(A + B) = \det(2I_2) = 2 \cdot 2 = 4.$$

But

$$\det(A) + \det(B) = 1 + 1 = 2.$$

Since

$$4 \neq 2,$$

we have

$$\det(A+B) \neq \det(A) + \det(B).$$

Therefore,

$$\boxed{\text{False}}.$$

(v) If three row interchanges are made in succession, then the new determinant equals the old determinant.

This statement is false.

Each row interchange multiplies the determinant by -1.

So after one row interchange, the determinant becomes

$$-\det(A).$$

After two row interchanges, it becomes

$$(-1)^2 \det(A) = \det(A).$$

After three row interchanges, it becomes

$$(-1)^3 \det(A) = -\det(A).$$

Therefore, after three row interchanges, the new determinant is the opposite of the old determinant, not the same.

So

$$\boxed{\text{False}}.$$

Final Answers for Exercise 4

$$\boxed{\text{(i) True}}$$

$$\boxed{\text{(ii) False}}$$

$$\boxed{\text{(iii) False}}$$

$$\boxed{\text{(iv) False}}$$

$$\boxed{\text{(v) False}}$$

Final Answer Summary

Exercise 1

$$\boxed{1(\text{a}) = 15}$$

$$\boxed{1(\text{b}) = 6}$$

Exercise 2

$$\boxed{2(\text{a}) = -19}$$

$$\boxed{2(\text{b}) = -16}$$

$$\boxed{2(\text{c}) = 0}$$

Exercise 3

$$\boxed{\det(AB) = -6}$$

$$\boxed{\det(5A) = -250}$$

$$\boxed{\det(B^T) = 3}$$

$$\boxed{\det(A^{-1}) = -\frac{1}{2}}$$

$$\boxed{\det(A^3) = -8}$$

Exercise 4

$$\boxed{\text{(i) True}}$$

$$\boxed{\text{(ii) False}}$$

$$\boxed{\text{(iii) False}}$$

$$\boxed{\text{(iv) False}}$$

$$\boxed{\text{(v) False}}$$

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