

Detailed Solutions: Eigenvalues, Eigenspaces, Characteristic Polynomials, and Diagonalization

Prepared for educational use
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Convention

For characteristic polynomials, I use the monic convention

$$p_A(\lambda) = \det(\lambda I - A).$$

If a course uses $\det(A - \lambda I)$, then for odd-dimensional matrices the polynomial differs by a factor of -1 . The characteristic equation and eigenvalues are unchanged.

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For each listed eigenvalue, the eigenspace is

$$E_\lambda = \ker(A - \lambda I).$$

9.

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \quad \lambda = 1, 5.$$

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 0$, while x_2 is free. Hence

$$E_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = 5$,

$$A - 5I = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 2x_2$. Hence

$$E_5 = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

10.

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \quad \lambda = 4.$$

$$A - 4I = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = \frac{3}{2}x_2$. Taking $x_2 = 2$ gives $x_1 = 3$. Therefore

$$E_4 = \text{span} \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}.$$

11.

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}, \quad \lambda = 10.$$

$$A - 10I = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = -\frac{1}{3}x_2$. Taking $x_2 = 3$ gives $x_1 = -1$. Therefore

$$E_{10} = \text{span} \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}.$$

12.

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \lambda = -2, 5.$$

For $\lambda = -2$,

$$A + 2I = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = -\frac{4}{3}x_2$. Taking $x_2 = 3$ gives

$$E_{-2} = \text{span} \left\{ \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}.$$

For $\lambda = 5$,

$$A - 5I = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus $x_1 = x_2$, so

$$E_5 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

13.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \lambda = 1, 2, 3.$$

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 0$, $x_3 = 0$, and x_2 is free:

$$E_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = -\frac{1}{2}x_3$ and $x_2 = x_3$. Taking $x_3 = 2$ gives

$$E_2 = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = -x_3$ and $x_2 = x_3$. Taking $x_3 = 1$ gives

$$E_3 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

14.

$$A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix}, \quad \lambda = -4.$$

$$A + 4I = \begin{bmatrix} 7 & -1 & 3 \\ -1 & 7 & 3 \\ 6 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = -\frac{1}{2}x_3$ and $x_2 = -\frac{1}{2}x_3$. Taking $x_3 = 2$ gives

$$E_{-4} = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

15.

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \quad \lambda = 3.$$

$$A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equation is $x_1 + 2x_2 + 3x_3 = 0$, so

$$x_1 = -2x_2 - 3x_3.$$

Taking $(x_2, x_3) = (1, 0)$ and $(0, 1)$, a basis is

$$E_3 = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

16.

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \lambda = 4.$$

$$A - 4I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 2x_3$, $x_2 = 3x_3$, and x_4 is also free. Therefore

$$E_4 = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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1.

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 2 & -7 \\ -7 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 49 = (\lambda - 9)(\lambda + 5).$$

The eigenvalues are

$$\lambda = 9, -5.$$

2.

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 5 & -3 \\ -3 & \lambda - 5 \end{bmatrix} = (\lambda - 5)^2 - 9 = (\lambda - 8)(\lambda - 2).$$

The eigenvalues are

$$\lambda = 8, 2.$$

3.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 3 & 2 \\ -1 & \lambda + 1 \end{bmatrix} = (\lambda - 3)(\lambda + 1) + 2 = \lambda^2 - 2\lambda - 1.$$

Solving $\lambda^2 - 2\lambda - 1 = 0$ gives

$$\lambda = 1 \pm \sqrt{2}.$$

4.

$$A = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 4 & 3 \\ 4 & \lambda - 2 \end{bmatrix} = (\lambda - 4)(\lambda - 2) - 12 = \lambda^2 - 6\lambda - 4.$$

Thus

$$\lambda = \frac{6 \pm \sqrt{36 + 16}}{2} = 3 \pm \sqrt{13}.$$

5.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 4 \end{bmatrix} = (\lambda - 2)(\lambda - 4) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

The eigenvalue is

$$\lambda = 3 \quad \text{with algebraic multiplicity 2.}$$

6.

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 1 & 4 \\ -4 & \lambda - 6 \end{bmatrix} = (\lambda - 1)(\lambda - 6) + 16 = \lambda^2 - 7\lambda + 22.$$

The eigenvalues are

$$\lambda = \frac{7 \pm \sqrt{49 - 88}}{2} = \frac{7 \pm i\sqrt{39}}{2}.$$

7.

$$A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 5 & -3 \\ 4 & \lambda - 4 \end{bmatrix} = (\lambda - 5)(\lambda - 4) + 12 = \lambda^2 - 9\lambda + 32.$$

The eigenvalues are

$$\lambda = \frac{9 \pm \sqrt{81 - 128}}{2} = \frac{9 \pm i\sqrt{47}}{2}.$$

8.

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 7 & 2 \\ -2 & \lambda - 3 \end{bmatrix} = (\lambda - 7)(\lambda - 3) + 4 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

The eigenvalue is

$$\lambda = 5 \quad \text{with algebraic multiplicity 2.}$$

9.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 1 & 0 & 1 \\ -2 & \lambda - 3 & 1 \\ 0 & -6 & \lambda \end{bmatrix}.$$

Expanding along the first row,

$$p_A(\lambda) = (\lambda - 1) \det \begin{bmatrix} \lambda - 3 & 1 \\ -6 & \lambda \end{bmatrix} + \det \begin{bmatrix} -2 & \lambda - 3 \\ 0 & -6 \end{bmatrix}.$$

Therefore

$$p_A(\lambda) = (\lambda - 1)(\lambda^2 - 3\lambda + 6) + 12 = \lambda^3 - 4\lambda^2 + 9\lambda + 6.$$

The eigenvalues are the roots of

$$\lambda^3 - 4\lambda^2 + 9\lambda + 6 = 0.$$

Approximate roots are

$$\lambda \approx -0.5270, \quad \lambda \approx 2.2635 \pm 2.5024i.$$

10.

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda & -3 & -1 \\ -3 & \lambda & -2 \\ -1 & -2 & \lambda \end{bmatrix}.$$

Expanding along the first row,

$$\begin{aligned} p_A(\lambda) &= \lambda \det \begin{bmatrix} \lambda & -2 \\ -2 & \lambda \end{bmatrix} + 3 \det \begin{bmatrix} -3 & -2 \\ -1 & \lambda \end{bmatrix} - \det \begin{bmatrix} -3 & \lambda \\ -1 & -2 \end{bmatrix} \\ &= \lambda(\lambda^2 - 4) + 3(-3\lambda - 2) - (6 + \lambda) \\ &= \lambda^3 - 14\lambda - 12. \end{aligned}$$

The eigenvalues are the roots of

$$\lambda^3 - 14\lambda - 12 = 0.$$

Approximate roots are

$$\lambda \approx -3.2019, \quad -0.9112, \quad 4.1131.$$

11.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 4 & 0 & 0 \\ -5 & \lambda - 3 & -2 \\ 2 & 0 & \lambda - 2 \end{bmatrix}.$$

Expanding along the first row,

$$p_A(\lambda) = (\lambda - 4) \det \begin{bmatrix} \lambda - 3 & -2 \\ 0 & \lambda - 2 \end{bmatrix} = (\lambda - 4)(\lambda - 3)(\lambda - 2).$$

The eigenvalues are

$$\lambda = 4, 3, 2.$$

12.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 1 & 0 & -1 \\ 3 & \lambda - 6 & -1 \\ 0 & 0 & \lambda - 4 \end{bmatrix}.$$

Expanding along the third row,

$$p_A(\lambda) = (\lambda - 4) \det \begin{bmatrix} \lambda - 1 & 0 \\ 3 & \lambda - 6 \end{bmatrix} = (\lambda - 4)(\lambda - 1)(\lambda - 6).$$

The eigenvalues are

$$\lambda = 4, 1, 6.$$

13.

$$A = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 6 & 2 & 0 \\ 2 & \lambda - 9 & 0 \\ -5 & -8 & \lambda - 3 \end{bmatrix}.$$

Expanding along the third column,

$$p_A(\lambda) = (\lambda - 3) \det \begin{bmatrix} \lambda - 6 & 2 \\ 2 & \lambda - 9 \end{bmatrix} = (\lambda - 3)((\lambda - 6)(\lambda - 9) - 4).$$

Thus

$$p_A(\lambda) = (\lambda - 3)(\lambda^2 - 15\lambda + 50) = (\lambda - 3)(\lambda - 5)(\lambda - 10).$$

The eigenvalues are

$$\lambda = 3, 5, 10.$$

14.

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 0 & -1 & 0 \\ 6 & 7 & -4 \end{bmatrix}.$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 3 & 2 & -3 \\ 0 & \lambda + 1 & 0 \\ -6 & -7 & \lambda + 4 \end{bmatrix}.$$

Expanding along the second row,

$$p_A(\lambda) = (\lambda + 1) \det \begin{bmatrix} \lambda - 3 & -3 \\ -6 & \lambda + 4 \end{bmatrix}.$$

Therefore

$$p_A(\lambda) = (\lambda + 1)((\lambda - 3)(\lambda + 4) - 18) = (\lambda + 1)(\lambda^2 + \lambda - 30) = (\lambda + 1)(\lambda - 5)(\lambda + 6).$$

The eigenvalues are

$$\lambda = -1, 5, -6.$$

15. The matrix is upper triangular:

$$A = \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a triangular matrix, the eigenvalues are the diagonal entries, repeated according to multiplicity. Hence

$$\lambda = 4, 3, 3, 1.$$

Equivalently,

$$p_A(\lambda) = (\lambda - 4)(\lambda - 3)^2(\lambda - 1).$$

16. The matrix is lower triangular:

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}.$$

Thus the eigenvalues are the diagonal entries:

$$\lambda = 5, -4, 1, 1.$$

Equivalently,

$$p_A(\lambda) = (\lambda - 5)(\lambda + 4)(\lambda - 1)^2.$$

17. The matrix is lower triangular:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}.$$

Thus the eigenvalues are the diagonal entries:

$$\lambda = 3, 1, 0, 1, 3.$$

Repeated according to multiplicity,

$$\lambda = 3, 3, 1, 1, 0.$$

Equivalently,

$$p_A(\lambda) = \lambda(\lambda - 3)^2(\lambda - 1)^2.$$

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For each diagonalization, the columns of P are eigenvectors in the same order as the eigenvalues on the diagonal of D . Then

$$A = PDP^{-1}.$$

7.

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}.$$

The eigenvalues are 1 and -1 . The eigenspaces are

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}, \quad E_{-1} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

So a diagonalization is

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus $A = PDP^{-1}$.

8.

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}.$$

The only eigenvalue is $\lambda = 5$, with algebraic multiplicity 2. Now

$$A - 5I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so

$$E_5 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The eigenspace has dimension 1, but a 2×2 matrix needs two linearly independent eigenvectors to be diagonalizable. Therefore A is not diagonalizable.

9.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}.$$

The characteristic polynomial is

$$p_A(\lambda) = (\lambda - 4)^2.$$

Thus $\lambda = 4$ has algebraic multiplicity 2. Also

$$A - 4I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$E_4 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace has dimension $1 < 2$, so A is not diagonalizable.

10.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues are 5 and -2 . The eigenspaces are

$$E_5 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad E_{-2} = \text{span} \left\{ \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

11.

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

The eigenvalues are 1, 2, 3. Eigenspace bases are

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad E_2 = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}, \quad E_3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

12.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

The eigenvalues are 1 and 4, where 4 has algebraic multiplicity 2. The eigenspaces are

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad E_4 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have three linearly independent eigenvectors, so A is diagonalizable. One diagonalization is

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

13.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

The eigenvalues are 5 and 1, where 1 has algebraic multiplicity 2. Eigenspace bases are

$$E_1 = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad E_5 = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Therefore

$$P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Thus $A = PDP^{-1}$.

14.

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

The eigenvalues are 3 and 4, where 3 has algebraic multiplicity 2. The eigenspaces are

$$E_3 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad E_4 = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

So

$$P = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

15.

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}.$$

The eigenvalues are 1 and 3, where 3 has algebraic multiplicity 2. Eigenspaces are

$$E_1 = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad E_3 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} -2 & -1 & -4 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

16.

$$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

The eigenvalues are 1 and 2, where 2 has algebraic multiplicity 2. Eigenspace bases are

$$E_1 = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad E_2 = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence

$$P = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

17.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues are 4 and 5, where 4 has algebraic multiplicity 2. Now

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$E_4 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Also

$$E_5 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The total number of linearly independent eigenvectors is only $1 + 1 = 2 < 3$. Therefore A is not diagonalizable.

18.

$$A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$p_A(\lambda) = (\lambda - 5)^2(\lambda + 3).$$

The vector $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = -3$, because

$$A \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Eigenspace bases are

$$E_{-3} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}, \quad E_5 = \text{span} \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} -2 & -4 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Therefore $A = PDP^{-1}$.

19.

$$A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Since A is upper triangular, the eigenvalues are 5, 3, 2, 2. Eigenspace bases are

$$E_2 = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad E_3 = \text{span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_5 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

There are four linearly independent eigenvectors, so A is diagonalizable. One diagonalization is

$$P = \begin{bmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Thus $A = PDP^{-1}$.

20.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

The only eigenvalue is $\lambda = 2$, with algebraic multiplicity 4. We compute

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_1 = 0$, while x_2, x_3, x_4 are free. Hence

$$E_2 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace dimension is $3 < 4$, so A is not diagonalizable.